Absolute (Global) Extreme Values

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DEFINITION **Absolute Extreme Values**

Let f be a function with domain D. Then $f(c)$ is the

(a) **absolute maximum value** on D if and only if $f(x) \leq f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \geq f(c)$ for all x in D.

EXAMPLE 1 **Exploring Extreme Values**

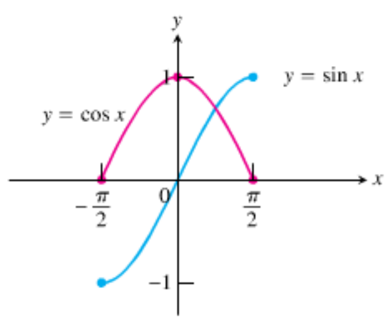
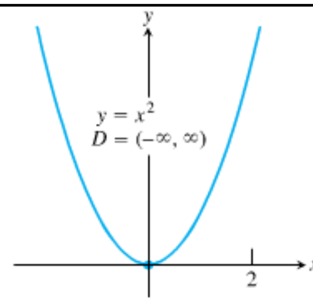
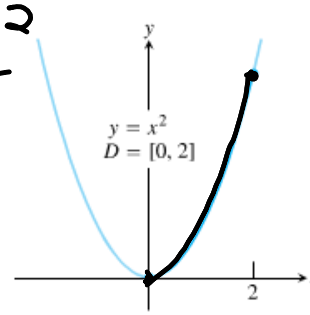


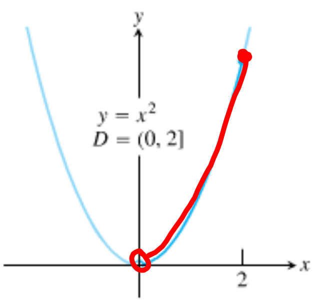
Figure 4.1 (Example 1)

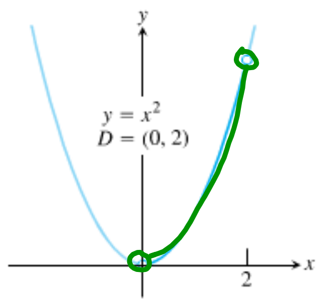
EXAMPLE 2 Exploring Absolute Extrema



	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	no max minimum of 0 at x = 0
(b)	$y = x^2$	$[0, 2]$	maximum of 4 at x = 2 min. of 0 at x = 0
(c)	$y = x^2$	$(0, 2]$	max of 4 at x = 2 no min
(d)	$y = x^2$	$(0, 2)$	no max no min







Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

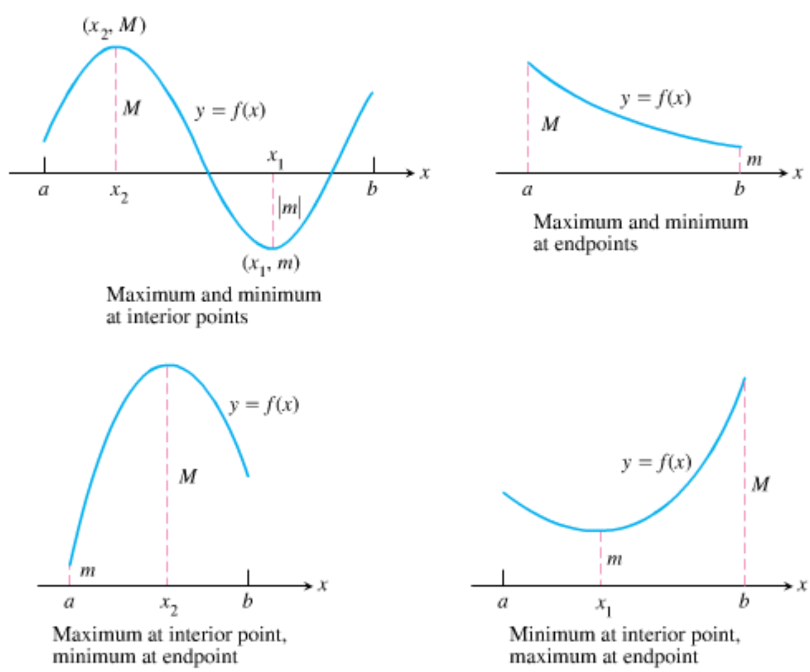


Figure 4.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

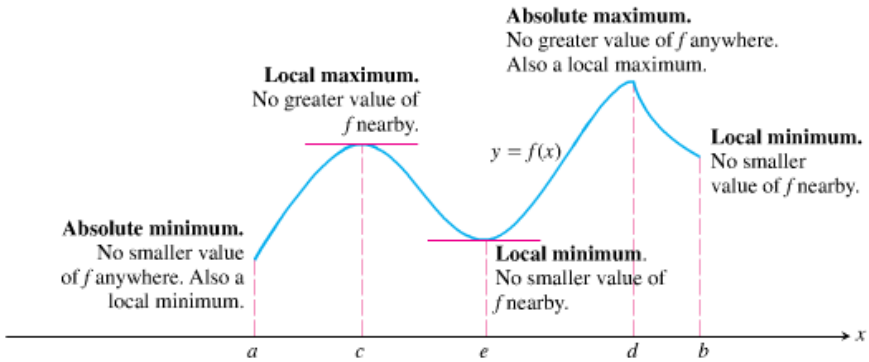


Figure 4.4 Classifying extreme values.

Local extrema are also called **relative extrema.**

DEFINITION Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) local maximum value at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) local minimum value at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

An **absolute extremum** is also a local extremum.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

DEFINITION Stationary Point

A point in the interior of the domain of a function f at which $f' = 0$ is called a ***stationary point*** of f.

EXAMPLE 3 Finding Absolute Extrema

domain $(-\infty, \infty)$

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$= \frac{2}{3\sqrt[3]{x}}$$

$f'(x)$ never $= 0$

$f'(x)$ does not exist when $x = 0$

critical value

endpoint values $f(-2) = \sqrt[3]{4}$

$f(3) = \sqrt[3]{9}$

critical value $f(0) = 0$

absolute max is $\sqrt[3]{9}$

absolute min is 0

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$. $\quad 4 - x^2 \geq 0$

domain: $(-2, 2)$

$$f(x) = (4 - x^2)^{-1/2}$$

$$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x)$$

$$= \frac{x}{\sqrt{(4 - x^2)^3}}$$

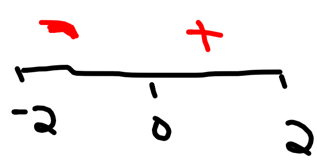
$f'(x) = 0$ when $x = 0$

$f'(x)$ always exists between -2 and 2

the only critical point

$f(0) = \frac{1}{2}$

minimum!

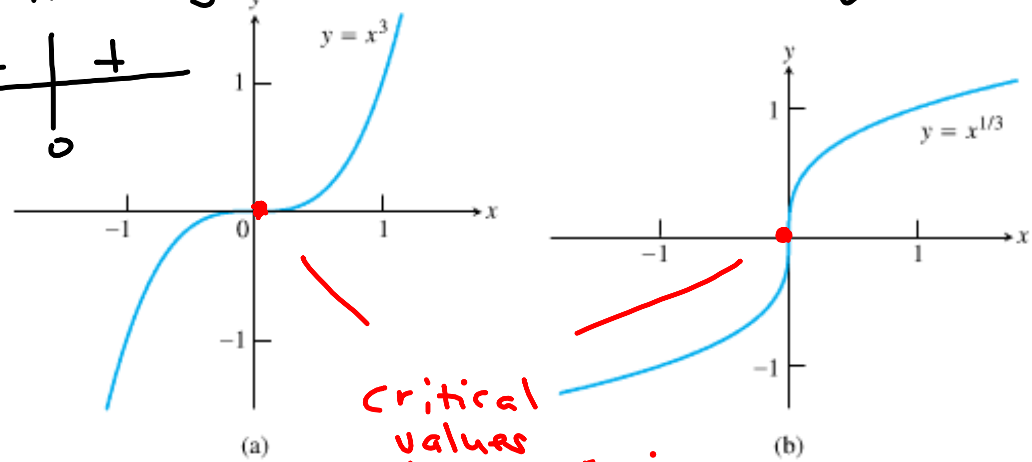


While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value.

no change in sign

+ + (0)

+ + (0) no change in sign



critical values
not max or min

Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

Is f(x) conitinuos at x = 1

$$\lim_{x \to 1^-} f(x) = 5 - 2(1)^2 = 3$$

$$\lim_{x \to 1^+} f(x) = 1 + 2 = 3 \qquad \text{yes}$$

$$f(1) = 5 - 2(1)^2 = 3$$

$$f'(x) \begin{cases} -4x & x < 1 \\ 1 & x > 1 \end{cases}$$

$+ \;\; | \;\; - \;\; | \;\; +$

$\quad\;\; 0 \qquad 1$

$f'(1)$ from left $\to -4$

$f'(1)$ from right $\to 1$

$f'(x)$ does not exist at $x = 1$

$f'(x) = 0$ when $x = 0$

$f(0) = 5 - 2(0)^2 = 5$ max

$f(1) = 5 - 2(1)^2 = 3$ min

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln\left|\dfrac{x}{1+x^2}\right|$.

SOLUTION